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Deformation Dynamics and the Gauss–Bonnet Topological Term in String Theory

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We show that there exists a nontrivial contribution on the Witten covariant phase space when the Gauss–Bonnet topological term is added to the Dirac–Nambu–Goto action describing strings, because the geometry of deformations is modified, and on such space we construct a symplectic structure. Future extensions of the present results are outlined.

KEY WORDS: symplectic structure; canonical quantization; strings.

1. INTRODUCTION

As we know, if we add the Gauss–Bonnet [GB] topological term in any action describing strings (for example the Dirac–Nambu–Goto action [DNG]), we do not find any contribution to the equations of motion, because the field equations of the [GB] topological term are proportional to the called Einstein tensor, and it does not give any contribution to the dynamics in a two-dimension worldsheet swept out by a string, since the Einstein tensor vanishes for such a geometry. In this manner, if we use the conventional canonical formalism based in the classical dynamics of the system, we would not find apparently nothing interesting.

However, using a covariant description of the canonical formalism (Crncović and Witten, 1987), and identifying the arguments of the total divergences at the level of the lagrangian as symplectic potentials (Escalante, 2004), in Cartas-Fuentevilla (2004) gives a sign that the [GB] topological term has a nontrivial contribution in the symplectic structure constructed on the classical covariant phase space. But, in Cartas-Fuentevilla (2004) important calculations are not developed, for example: the contribution of the [GB] topological term to the linearized equations of motion that are useful for stability analysis, subsequently the construction of a covariantly conserved symplectic current that allows us to establish a conection between functions and Hamiltonian vector fields, and the correct identification of the exterior derivative on the phase space.

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In this manner, the purpose of this article is to show in a clear way how the [GB] topological term modifies the symplectic geometry of the Witten covariant phase space, when we add it to the [DNG] action for string theory, by developing new ideas and completing the results presented in Cartas-Fuentevilla (2004).

This paper is organized as follows. In the Sect. 2, using the deformations formalism introduced in Capovilla and Guven (1995), we calculate the normal and tangential deformations of quantities on the embedding, that will be useful in our developments. In Sect. 3, we obtain the equations of motion and identify the corresponding symplectic potential for [DNG-GB] p-branes, also we calculate the linearized equations of motion and show that in general the [GB] topological term gives indeed a nontrivial contribution on the deformations geometry. In Sect. 3.1, we take the results obtained in Sect. 3 for the case of string theory, and we show that in spite of the dynamics for [DNG] and [DNG-GB] in string theory being the same, the simplectic potential, and the linearized dynamics are modified because of the [GB] topological term, and therefore, there is a relevant contribution on the phase space. In Sect. 4, from the linearized equations obtained in Sect. 3.1 for string theory, we obtain a symplectic current by applying the self-adjoint operators scheme, proving that there is a world sheet covariantly conserved current. In Sect. 5, we define the Witten covariant phase space for [DNG-GB] strings, and using the symplectic current found in Sect. 4, we construct a geometrical structure, showing that it has a relevant contribution because of the [GB] topological term. In Sect. 6, we give conclusions and prospects.

2. DEFORMATIONS OF THE EMBEDDING

In the scheme of deformations (Capovilla and Guven, 1995), the physically observable measure of the deformations of the embedding is given by the orthogonal projection of the infinitesimal spacetime variations $\delta X^{\mu} = n_i{}^{\mu}\phi^i$, and the tangential deformations together with the total divergence terms are neglected. However, in Escalante (2004) it is shown that tangential deformations and divergence terms are important because such terms are identified as symplectic potentials, whose variations (the exterior derivative on the space phase) generate the integral kernel of a covariant and gauge invariant symplectic structure, for the theory under study. Thus, for a complete analysis, we do not only need to calculate the normal deformations of fields defined on the embedding as in Capovilla and Guven (1995), but also the tangential deformations, that will be important in the development of this work.

For this purpose, we decompose an arbitrary infinitesimal deformation of the embedding δX^{μ} into its parts tangential and normal to the worldsheet

$$\delta X^{\mu} = e_a{}^{\mu}\phi^a + n_i{}^{\mu}\phi^i, \tag{1}$$

where ${n_i}^{\mu}$ are vector fields normal to the worldsheet and ${e_a}^{\mu}$ are vector fields tangent to such a worldsheet, thus, the deformation operator is defined as

$$\mathbf{D} = D_\delta + D_\Delta, \tag{2}$$

where

$$D_\delta = \delta^\mu D_\mu, \quad \delta^\mu = {n_i}^\mu \phi^i, \tag{3}$$

and

$$D_\Delta = \Delta^\mu D_\mu, \quad \Delta^\mu = {e_a}^\mu \phi^a, \tag{4}$$

therefore, we can find that the deformations of the intrinsic geometry of the embedding are given by

$$\mathbf{D}e_a = \left({K_{ab}}^i \phi_i\right)e^b + (\widetilde{\nabla}_a \phi_i)n^i + (\nabla_a \phi^b)e_b - {K_{ab}}^i \phi^b n_i, \tag{5}$$

$$\mathbf{D}\gamma_{ab} = 2{K_{ab}}^j \phi_j + \nabla_a \phi_b + \nabla_b \phi_a, \tag{6}$$

$$\mathbf{D}\gamma^{ab} = -2K^{abj}\phi_j - \nabla^a \phi^b - \nabla^b \phi^a, \tag{7}$$

$$\mathbf{D}\sqrt{-\gamma} = \sqrt{-\gamma}[\nabla_a \phi^a + K^i \phi_i], \tag{8}$$

$$\begin{aligned}\mathbf{D}{\gamma_{gf}}^a = {} & \gamma^{ad}\left[\nabla_f\left({K_{gd}}^j \phi_j\right) + \nabla_g\left({K_{fd}}^j \phi_j\right) - \nabla_d\left(\left({K_{gf}}^j \phi_j\right)\right)\right] \\ & + \frac{1}{2}\gamma^{ad}\left[2\nabla_{(g}\nabla_{f)}\phi_d - {R^e}_{fdg}\phi_e - {R^e}_{gdf}\phi_e\right],\end{aligned} \tag{9}$$

$$\mathbf{D}R_{ab} = \nabla_c\left(\mathbf{D}{\gamma_{ab}}^c\right) - \nabla_b\left(\mathbf{D}{\gamma_{ac}}^c\right), \tag{10}$$

$$\begin{aligned}\mathbf{D}R = {} & (\mathbf{D}\gamma^{ab})R_{ab} + \gamma^{ab}(\mathbf{D}R_{ab}) = -2\nabla^a \phi^b R_{ab} - 2K^{abj}\phi_j R_{ab} \\ & + \gamma^{ab}\left[\nabla_c\left(\mathbf{D}{\gamma_{ab}}^c\right) - \nabla_b\left(\mathbf{D}{\gamma_{ac}}^c\right)\right],\end{aligned} \tag{11}$$

where, ${K_{ab}}^i$, γ_{ab}, ${\gamma_{gf}}^a$, R, R_{ab} are the extrinsic curvature, the metric, the connection coefficients, the scalar curvature, and the Ricci tensor of the worldsheet, respectively (Capovilla and Guven, 1995).

For this article this is sufficient on the deformations of embedding.

3. THE DNG ACTION FOR p-BRANES WITH A GAUSS–BONNET TERM

As we know, the [DNG] action for p-branes is proportional to the area of the spacetime trajectory created by the brane, and the Gauss–Bonnet term is proportional to the Ricci scalar R constructed from the world surface metric γ_{ab}. Both terms are given in the following action

$$S = -\sigma \int \sqrt{-\gamma}\, d^D\xi + \beta \int \sqrt{-\gamma}\, R d^D \xi, \tag{12}$$

where σ and β are constants, and D is the dimension of the worldsheet.

In agreement with Eqs. (8)–(11), the variation of the action (12) is given by

$$\mathbf{D}S = -\int \sqrt{-\gamma}\big[\sigma K^i + 2\beta G_{ab}K^{abi}\big]\phi_i d^D\xi$$
$$+\int \sqrt{-\gamma}\nabla_a\big[-\sigma\phi^a - 2\beta G^{ab}\phi_b + \beta\gamma^{cd}\mathbf{D}\gamma^a_{cd} - \beta\gamma^{ab}\mathbf{D}\gamma_{cb}{}^c\big]d^D\xi, \tag{13}$$

where we can identify the equations of motion

$$\sigma K^i + 2\beta G_{ab}K^{abi} = 0, \tag{14}$$

. being G_{ab} the world surface Einstein tensor given by

$$G_{ab} = R_{ab} - \frac{1}{2}\gamma_{ab}R, \tag{15}$$

and the argument of the pure divergence term is identified as a symplectic potential for the theory (Escalante, 2004), as it will be proved below.

$$\Psi^a = \sqrt{-\gamma}\big[-\sigma\phi^a - 2\beta G^{ab}\phi_b + \beta\gamma^{cd}\mathbf{D}\gamma^a_{cd} - \beta\gamma^{ab}\mathbf{D}\gamma_{cb}{}^c\big]. \tag{16}$$

We notice from Eq. (16), that the symplectic potential found in Cartas-Fuentevilla (2004) is incomplete; the reason is that the normal variation (D_δ) is considered as exterior derivative on the phase space, however, as we will show in the next sections the correct exterior derivative on the phase space is the sum of normal and tangential variations ($D_\delta + D_\Delta$).

On the other hand, such as in Escalante (2004), the variation of Ψ^a (the derivative exterior on the phase space) will generate the integral kernel of a covariant and gauge invariant symplectic structure for [DNG-GB] theory. In this manner, we can see in Eq. (16) that in general there is a relevant contribution on the phase space because of the terms proportional to the parameter β, coming from the [GB] term.

In order to give a more detailed analysis, let us see how the [GB] term contribute to the linearized equations of motion when we add it to the [DNG] action for p-branes, for this we calculate the variations of the Eq. (14), obtaining

$$\begin{aligned}
&\sigma[-\widetilde{\Delta}^i_j - K_{ab}{}^i K^{ab}{}_j + g(\mathbf{R}(e_a, n_j)e^a, n^i)]\phi^j \\
&+ 2\beta G^{ab}[-\widetilde{\nabla}_a\widetilde{\nabla}_b\phi^i + K_{ad}{}^i K^d{}_b{}^j\phi_j + g(\mathbf{R}(e_a, n^j)e_b, n^i)\phi_j] \\
&- 8\beta K^b{}_d{}^i K^{adj}\phi_j G_{ab} + 4\beta K^{abi}\widetilde{\nabla}_c\widetilde{\nabla}_b K_a{}^{cj}\phi_j + 4\beta K^{abi}\widetilde{\nabla}_b K_a{}^{cj}\widetilde{\nabla}_c\phi_j \\
&+ 4\beta K^{abi}\widetilde{\nabla}_c K_a{}^{cj}\widetilde{\nabla}_b\phi_j + 4\beta K^{abi}K_a{}^{cj}\widetilde{\nabla}_c\widetilde{\nabla}_b\phi_j - 2\beta K^{abi}\widetilde{\Delta}K_{ab}{}^j\phi_j \\
&- 4\beta K^{abi}\widetilde{\nabla}_c K^{abj}\widetilde{\nabla}^c\phi_j - 2\beta K^{abi}K_{ab}{}^j\widetilde{\Delta}\phi_j - 2\beta K^{abi}\widetilde{\nabla}_b\widetilde{\nabla}_a K^j\phi_j \\
&- 4\beta K^{abi}\widetilde{\nabla}_a K^j\widetilde{\nabla}_b\phi_j - 2\beta K^{abi}K^j\widetilde{\nabla}_b\widetilde{\nabla}_a\phi_j - 2\beta K^{abi}K_{ab}{}^j\phi^j R
\end{aligned}$$

$$
\begin{aligned}
&+2\beta R_{cd}K^{cd}{}_{j}\phi^{j}K^{i}+2\beta K^{i}\widetilde{\Delta}K^{j}\phi_{j}+4\beta K^{i}\widetilde{\nabla}_{c}K^{j}\widetilde{\nabla}^{c}\phi_{j}\\
&+2\beta K^{i}K^{j}\widetilde{\Delta}\phi_{j}-2\beta K^{i}\widetilde{\nabla}_{c}\widetilde{\nabla}_{g}K^{gcj}\phi_{j}-4\beta K^{i}\widetilde{\nabla}_{g}K^{gcj}\widetilde{\nabla}_{c}\phi_{j}\\
&-2\beta K^{i}K^{cgj}\widetilde{\nabla}_{c}\widetilde{\nabla}_{g}\phi_{j}=0, \qquad (17)
\end{aligned}
$$

where $g(\mathbf{R}(Y_1, Y_2,)Y_3, Y_4) \equiv \mathbf{R}_{\alpha\beta\gamma\nu}Y_2{}^{\alpha}Y_1{}^{\beta}Y_3{}^{\gamma}Y_4{}^{\nu}$, $\mathbf{R}_{\alpha\beta\gamma\nu}$ being the background Riemman tensor (Capovilla and Guven, 1995). We can identify the first term of the last equation as the linearized dynamics of [DNG] theory, and the proportional terms to the parameter β as the contribution of [GB] term. As one would expect, if the parameter β vanished, we should obtain the linearized equations for [DNG] theory (Capovilla and Guven, 1995; Cartas-Fuentevilla, 2002). Thus, we can see that in general there is a relevant contribution to linearized equations because of [GB] term when we add it in the [DNG] action for p-branes.

3.1. The DNG Action With a Gauss–Bonnet Topological Term in Closed String Theory

In this section we will see what happens when we consider in Eqs. (14), (16), and (17), the case of string theory. For this, we know that in a two-dimensional worldsheet surface, swept out for a string, the Einstein tensor vanishes $G_{ab} = 0$, and Eq. (14) takes the form

$$K^{i} = 0, \qquad (18)$$

where we can see that the dynamics for [DNG] and [DNG-GB] in string theory are the same, and we do not find any contribution because of [GB] topological term, thus, if we use a conventional formulation to quantize the [DNG-GB] strings from corresponding classical dynamics (Eq. (18)) the same result is obtained and we would not find apparently any interest for including the [GB] topological term in any action describing strings. However, when in Eq. (16) we consider the case of string theory we obtain

$$\Psi^{a} = \sqrt{-\gamma}\big[-\sigma\phi^{a} + \beta\gamma^{cd}\mathbf{D}\gamma^{a}_{cd} - \beta\gamma^{ab}\mathbf{D}\gamma_{cb}{}^{c}\big]. \qquad (19)$$

In this manner, we can see that the last two terms correspond to the topological term that do not vanish and give a nontrivial contribution to the phase space as we will see in the next sections.

For the purposes of this paper, we take the case of string theory in Eq. (17), obtaining

$$
\begin{aligned}
&\sigma[-\widetilde{\Delta}^{i}_{j}-K_{ab}{}^{i}K^{ab}{}_{j}+g(\mathbf{R}(e_a, n_j)e^{a}, n^{i})]\phi^{j}\\
&+4\beta K^{abi}\widetilde{\nabla}_{c}\widetilde{\nabla}_{b}K_{a}{}^{cj}\phi_{j}+4\beta K^{abi}\widetilde{\nabla}_{b}K_{a}{}^{cj}\widetilde{\nabla}_{c}\phi_{j}+4\beta K^{abi}\widetilde{\nabla}_{c}K_{a}{}^{cj}\widetilde{\nabla}_{b}\phi_{j}\\
&+4\beta K^{abi}K_{a}{}^{cj}\widetilde{\nabla}_{c}\widetilde{\nabla}_{b}\phi_{j}-2\beta K^{abi}\widetilde{\Delta}K_{ab}{}^{j}\phi_{j}-4\beta K^{abi}\widetilde{\nabla}_{c}K^{abj}\widetilde{\nabla}^{c}\phi_{j}
\end{aligned}
$$

$$
\begin{aligned}
&-2\beta K^{abi}K_{ab}{}^{j}\widetilde{\Delta}\phi_j - 2\beta K^{abi}\widetilde{\nabla}_b\widetilde{\nabla}_a K^j\phi_j - 4\beta K^{abi}\widetilde{\nabla}_a K^j\widetilde{\nabla}_b\phi_j\\
&-2\beta K^{abi}K^j\widetilde{\nabla}_b\widetilde{\nabla}_a\phi_j - 2\beta K^{abi}K_{ab}{}^{j}\phi^j R + 2\beta R_{cd}K^{cd}{}_j\phi^j K^i\\
&+2\beta K^i\widetilde{\Delta}K^j\phi_j + 4\beta K^i\widetilde{\nabla}_c K^j\widetilde{\nabla}^c\phi_j + 2\beta K^i K^j\widetilde{\Delta}\phi_j - 2\beta K^i\widetilde{\nabla}_c\widetilde{\nabla}_g K^{gcj}\phi_j\\
&-4\beta K^i\widetilde{\nabla}_g K^{gcj}\widetilde{\nabla}_c\phi_j - 2\beta K^i K^{cgj}\widetilde{\nabla}_c\widetilde{\nabla}_g\phi_j = 0,
\end{aligned}
\tag{20}
$$

where we can see that there is also a contribution to the [DNG]'s linearized equations because of [GB] topological term, which is completely unknown in the literature. It is remarkable to mention that the linearized equations for [DNG-GB] strings theory, Eq. (20), can be useful in stability analysis, however, this is far from our purposes and we shall leave it as an open question, and we shall focus on the effects of the [GB] topological term on the phase space.

On the other hand, we consider the Eq. (18) in Eq. (20), obtaining

$$
P^{ij}\phi_j = 0 \tag{21}
$$

where the operator P^{ij} is given for

$$
\begin{aligned}
P^{ij} = \big[&\sigma\big\{-\widetilde{\Delta}^{ij} - K_{ab}{}^{i}K^{abj} + g(R(e_a, n^j)e^a, n^i)\big\} + 4\beta K^{abi}\widetilde{\nabla}_c\widetilde{\nabla}_b K_a{}^{cj}\\
&+4\beta K^{abi}\widetilde{\nabla}_b K_a{}^{cj}\widetilde{\nabla}_c + 4\beta K^{abi}\widetilde{\nabla}_c K_a{}^{cj}\widetilde{\nabla}_b + 4\beta K^{abi}K_a{}^{cj}\widetilde{\nabla}_c\widetilde{\nabla}_b\\
&-2\beta K^{abi}\widetilde{\Delta}K_{ab}{}^{j} - 4\beta K^{abi}\widetilde{\nabla}_c K^{abj}\widetilde{\nabla}^c\\
&-2\beta K^{abi}K_{ab}{}^{j}\widetilde{\Delta} - 2\beta K^{abi}K_{ab}{}^{j}R\big].
\end{aligned}
\tag{22}
$$

In the next section we will apply the self-adjoint operators method to Eq. (21), and we will demonstrate that the operator P^{ij} given in Eq. (22) is self-adjoint, obtaining a symplectic current from this property.

4. SELF-ADJOINTNESS OF THE LINEARIZED DYNAMICS

In this section we shall demonstrate that the operator $P^i{}_j$, given in Eq. (22), is indeed self-adjoint and in this manner we shall construct a symplectic current in terms of solutions of the Eq. (21). With this purpose, let $\phi_1{}^i$ and $\phi_2{}^i$ be two arbitrary scalar fields, which correspond to a pair of solutions of the Eq. (21), thus we can verify the following:

$$
-\sigma\phi_{1i}\widetilde{\Delta}\phi_2{}^i = -\sigma\widetilde{\Delta}\phi_{1i}\phi_2{}^i + \nabla_a j_1{}^a, \tag{23}
$$

$$
\begin{aligned}
4\beta K^{abi}\widetilde{\nabla}_c\widetilde{\nabla}_b K_a{}^{cj}\phi_{1i}\phi_{2j} &= 4\beta K^{abi}\widetilde{\nabla}_c\widetilde{\nabla}_b K_a{}^{cj}\phi_{1i}\phi_{2j}\\
&\quad -4\beta K^{abj}\widetilde{\nabla}_c\widetilde{\nabla}_b K_a{}^{ci}\phi_{1i}\phi_{2j}\\
&\quad +4\beta K^{abj}\widetilde{\nabla}_c\widetilde{\nabla}_b K_a{}^{ci}\phi_{1i}\phi_{2j},
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
4\beta K^{abi}\widetilde{\nabla}_b K_a{}^{cj}\phi_{1i}\widetilde{\nabla}_c\phi_{2j} =& -4\beta\widetilde{\nabla}_a K^{bci}\widetilde{\nabla}_b K_c{}^{aj}\phi_{1i}\phi_{2j} \\
& -4\beta K^{bci}\widetilde{\nabla}_a\widetilde{\nabla}_b K_c{}^{aj}\phi_{1i}\phi_{2j} \\
& -4\beta K^{bci}\widetilde{\nabla}_b K_c{}^{aj}\widetilde{\nabla}_a\phi_{1i}\phi_{2j} + \nabla_a j_2{}^a, \qquad (25)\\
4\beta K^{abi}\widetilde{\nabla}_c K_b{}^{cj}\phi_{1i}\widetilde{\nabla}_a\phi_{2j} =& -4\beta\widetilde{\nabla}_a K^{abi}\widetilde{\nabla}_c K_b{}^{cj}\phi_{1i}\phi_{2j} \\
& -4\beta K^{abi}\widetilde{\nabla}_a\widetilde{\nabla}_c K_b{}^{cj}\phi_{1i}\phi_{2j} \\
& -4\beta K^{abi}\widetilde{\nabla}_c K_b{}^{cj}\widetilde{\nabla}_a\phi_{1i}\phi_{2j} + \nabla_a j_3{}^a, \qquad (26)\\
4\beta K^{abi}K_a{}^{cj}\phi_{1i}\widetilde{\nabla}_c\widetilde{\nabla}_b\phi_{2j} =& \ 4\beta\widetilde{\nabla}_a\widetilde{\nabla}_b K^{cai}K_c{}^{bj}\phi_{1i}\phi_{2j} \\
& +4\beta\widetilde{\nabla}_b K^{cai}\widetilde{\nabla}_a K_c{}^{bj}\phi_{1i}\phi_{2j} \\
& +4\beta\widetilde{\nabla}_b K^{cai}K_c{}^{bj}\widetilde{\nabla}_a\phi_{1i}\phi_{2j} \\
& +4\beta\widetilde{\nabla}_a K^{cai}\widetilde{\nabla}_b K_c{}^{bj}\phi_{1i}\phi_{2j} \\
& +4\beta K^{cai}\widetilde{\nabla}_a\widetilde{\nabla}_b K_c{}^{bj}\phi_{1i}\phi_{2j} \\
& +4\beta K^{cai}\widetilde{\nabla}_b K_c{}^{bj}\widetilde{\nabla}_a\phi_{1i}\phi_{2j} \\
& +4\beta\widetilde{\nabla}_a K^{cai}K_c{}^{bj}\widetilde{\nabla}_b\phi_{1i}\phi_{2j} \\
& +4\beta K^{cai}\widetilde{\nabla}_a K_c{}^{bj}\widetilde{\nabla}_b\phi_{1i}\phi_{2j} \\
& +4\beta K^{cai}K_c{}^{bj}\widetilde{\nabla}_a\widetilde{\nabla}_b\phi_{1i}\phi_{2j} + \nabla_a j_4{}^a, \qquad (27)\\
-2\beta K^{abi}\widetilde{\Delta}K_{ab}{}^j\phi_{1i}\phi_{2j} =& -2\beta K^{abi}\widetilde{\Delta}K_{ab}{}^j\phi_{1i}\phi_{2j} + 2\beta K^{abj}\widetilde{\Delta}K_{ab}{}^i\phi_{1i}\phi_{2j} \\
& -2\beta K^{abj}\widetilde{\Delta}K_{ab}{}^i\phi_{1i}\phi_{2j}, \qquad (28)\\
-4\beta K^{cdi}\widetilde{\nabla}_a K_{cd}{}^j\phi_{1i}\widetilde{\nabla}^a\phi_{2j} =& \ 4\beta\widetilde{\nabla}^a K^{cdi}\widetilde{\nabla}_a K_{cd}{}^j\phi_{1i}\phi_{2j} + 4\beta K^{cdi}\widetilde{\Delta}K_{cd}{}^j\phi_{1i}\phi_{2j} \\
& +4\beta K^{cdi}\widetilde{\nabla}_a K_{cd}{}^j\widetilde{\nabla}^a\phi_{1i}\phi_{2j} + \nabla_a j_5{}^a, \qquad (29)\\
-2\beta K^{cdi}K_{cd}{}^j\phi_{1i}\widetilde{\Delta}\phi_{2j} =& -2\beta\widetilde{\Delta}K^{cdi}K_{cd}{}^j\phi_{1i}\phi_{2j} \\
& -4\beta\widetilde{\nabla}_a K^{cdi}\widetilde{\nabla}^a K_{cd}{}^j\phi_{1i}\phi_{2j} \\
& -4\beta\widetilde{\nabla}_a K^{cdi}K_{cd}{}^j\widetilde{\nabla}^a\phi_{1i}\phi_{2j} \\
& -2\beta K^{cdi}\widetilde{\Delta}K_{cd}{}^j\phi_{1i}\phi_{2j} \\
& -4\beta K^{cdi}\widetilde{\nabla}_a K_{cd}{}^j\widetilde{\nabla}^a\phi_{1i}\phi_{2j} \\
& -2\beta K^{cdi}K_{cd}{}^j\widetilde{\Delta}\phi_{1i}\phi_{2j} + \nabla_a j_6{}^a, \qquad (30)
\end{aligned}
$$

where

$$j_1{}^a = \sigma\left[-\phi_{1i}\widetilde{\nabla}^a\phi_2{}^i + \widetilde{\nabla}^a\phi_{1i}\phi_2{}^i\right], \qquad (31)$$

$$j_2{}^a = 4\beta K^{bci}\widetilde{\nabla}_b K_c{}^{aj}\phi_{1i}\phi_{2j}, \qquad (32)$$

$$j_3{}^a = 4\beta K^{abi}\widetilde{\nabla}_c K_b{}^{cj}\phi_{1i}\phi_{2j}, \tag{33}$$

$$\begin{aligned} j_4{}^a = \beta\big[&4K^{cbi}K_c{}^{aj}\phi_{1i}\widetilde{\nabla}_b\phi_{2j} - 4\widetilde{\nabla}_b K^{cai}K_c{}^{bj}\phi_{1i}\phi_{2j} \\ &- 4K^{cai}\widetilde{\nabla}_b K_c{}^{bj}\phi_{1i}\phi_{2j} - 4K^{cai}K_c{}^{bj}\widetilde{\nabla}_b\phi_{1i}\phi_{2j}\big], \end{aligned} \tag{34}$$

$$j_5{}^a = -4\beta K^{cdi}\widetilde{\nabla}^a K_{cd}{}^j\phi_{1i}\phi_{2j} \tag{35}$$

$$\begin{aligned} j_6{}^a = \beta\big[&-2K^{cdi}K_{cd}{}^j\phi_{1i}\widetilde{\nabla}^a\phi_{2j} + 2\widetilde{\nabla}^a K^{cdi}K_{cd}{}^j\phi_{1i}\phi_{2j} \\ &+ 2K^{cdi}\widetilde{\nabla}^a K_{cd}{}^j\phi_{1i}\phi_{2j} + 2K^{cdi}K_{cd}{}^j\widetilde{\nabla}^a\phi_{1i}\phi_{2j}\big]. \end{aligned} \tag{36}$$

In this manner, considering the Eqs. (22)–(35), and after some arrangements, we obtain

$$\phi_{1i}(P^{ij})\phi_{2j} = (P^{ji})\phi_{1i}\phi_{2j} + \nabla_a j^a, \tag{37}$$

where $j^a = \Sigma_{i=1}^6 j_i{}^a$, which we can simplifly by substituting explicitly the Eqs. (31)–(36):

$$\begin{aligned} j^a = \sigma\big[&-\phi_{1i}\widetilde{\nabla}^a\phi_2{}^i + \widetilde{\nabla}^a\phi_{1i}\phi_2{}^i\big] + 4\beta K^{bci}\widetilde{\nabla}_b K_c{}^{aj}\phi_{1i}\phi_{2j} \\ &- 4\beta K^{cdi}\widetilde{\nabla}^a K_{cd}{}^j\phi_{1i}\phi_{2j} + \beta\big[4K^{cbi}K_c{}^{aj}\phi_{1i}\widetilde{\nabla}_b\phi_{2j} \\ &- 4\widetilde{\nabla}_b K^{cai}K_c{}^{bj}\phi_{1i}\phi_{2j} - 4K^{cai}K_c{}^{bj}\widetilde{\nabla}_b\phi_{1i}\phi_{2j}\big] \\ &+ \beta\big[-2K^{cdi}K_{cd}{}^j\phi_{1i}\widetilde{\nabla}^a\phi_{2j} + 2\widetilde{\nabla}^a K^{cdi}K_{cd}{}^j\phi_{1i}\phi_{2j} \\ &+ 2K^{cdi}\widetilde{\nabla}^a K_{cd}{}^j\phi_{1i}\phi_{2j} + 2K^{cdi}K_{cd}{}^j\widetilde{\nabla}^a\phi_{1i}\phi_{2j}\big]. \end{aligned} \tag{38}$$

In this manner, Eq. (37) implies that the operator P^{ij} is self-adjoint, and considering that ϕ_{1i}, and ϕ_{2j} correspond to solutions of the Eq. (21) ($P^{ij}\phi_{1j} = P^{ij}\phi_{2j} = 0$), j^a given in Eq. (38) is a worldsheet covariantly conserved

$$\nabla_a j^a = 0. \tag{39}$$

In the next section, we will compare the expression Eq. (38) with the variation of the symplectic potential given in Eq. (19) on the phase space, and we will demostrate that they are exactly the same.

5. THE WITTEN PHASE SPACE FOR [DNG-GB] STRINGS AND THE SYMPLECTIC STRUCTURE ON *Z*

In accordance with Crncović and Witten (1987), in a given physical theory, the classical phase space is the space of solutions of the classical equations of motion, which corresponds to a manifestly covariant definition, and on such phase space we can construct a covariant and gauge invariant symplectic structure. The basic idea to construct a symplectic structure on the space phase is to describe Poisson brackets of the theory in terms of it, instead of choosing p's and q's.

Based on the last paragraph, the Witten phase space for [DNG-GB] p-branes, is the space of solutions of Eq. (14)

$$\sigma K^i + 2\beta G_{ab} K^{abi} = 0,$$

but, for [DNG-GB] strings $G_{ab} = 0$ is the set of solutions of Eq. (18)

$$K^i = 0,$$

that we shall call Z, and on this phase space we will construct a symplectic structure. We can notice that the phase space for [DNG] strings (Cartas-Fuentevilla, 2002; Escalante, 2004) is the same for [DNG-GB] strings, Eq. (18), but the corresponding symplectic structures, that are in the transition of the regimens classical and quantum, will be different, as we shall see below. In the literature, using a conventional canonical formulation to quantize the [DNG-GB] strings from the corresponding classical dynamics, Eq. (18), the same results are obtained whether we include the topological term or not, but in this scheme of quantization there is an important contribution of such term as in path integral formalism, where such term has a relevant contribution weighting the different topologies in the sum over world surfaces.

Thus, following to Cartas-Fuentevilla (2002) and Escalante (2004), we can identify the scalar fields ϕ^i, ϕ^a as one-forms on Z and therefore anticommutating objects: $\phi^i\phi^j = -\phi^j\phi^i$, and $\phi^a\phi^b = -\phi^b\phi^a$. Additionally, in Cartas-Fuentevilla (2002, 2004) the vector field $\delta = n^i\phi_i$ is identified as the exterior derivative on Z, but it is incomplete, because the exterior derivative on Z changes when we consider the importance of the tangential deformations, and becomes

$$\delta = n^i\phi_i + e^a\phi_a, \tag{40}$$

since it is the correct exterior derivative which satisfies

$$\delta^2 = (n^i\phi_i + e^a\phi_a)(n^j\phi_j + e^b\phi_b) = 0, \tag{41}$$

which vanishes because of the commutativity of the zero-forms n^i, e^a and the anticommutativity of the one-forms ϕ^i, ϕ^a on Z (Cartas-Fuentevilla, 2002; Escalante, 2004).

In this manner, if we calculate the variation of symplectic potential given in Eq. (19), we obtain

$$\delta\Psi^a = \mathbf{D}\Psi^a = \sqrt{-\gamma}\, j'^a, \tag{42}$$

with j'^a given by

$$\begin{aligned} j'^a = {}& \sigma\phi_i\widetilde{\nabla}^a\phi^i - 4\beta K^{bci}\widetilde{\nabla}_b K_c{}^{aj}\phi_i\phi_j - 4\beta K^{bci}K_c{}^{aj}\phi_i\widetilde{\nabla}_b\phi_j \\ & + 2\beta K^{cdi}K_{cd}{}^j\phi_i\widetilde{\nabla}^a\phi_j + 2\beta K^{cdi}\widetilde{\nabla}^a K_{cd}{}^j\phi_i\phi_j, \end{aligned} \tag{43}$$

where we have used Eqs. (8), (9), and (41), and we have gauged away the ϕ^a terms, because it is identified as a diffeomorphism on the worldsheet (Escalante, 2004).

Now, we compare the two-form obtained in Eq. (43) with the symplectic current found in last section considering the self-adjointness of the linearized dynamics. For this purpose, we can set $\phi_{1i} = \phi_{2i} = \phi_i$, and use the antisymmetry of this scalar field in Eq. (38) (Cartas-Fuentevilla, 2002; Escalante, 2004), to obtain

$$\begin{aligned} j^a &= \sigma\phi_i\widetilde{\nabla}^a\phi^i - 4\beta K^{bci}\widetilde{\nabla}_b K_c{}^{aj}\phi_i\phi_j - 4\beta K^{bci}K_c{}^{aj}\phi_i\widetilde{\nabla}_b\phi_j \\ &\quad + 2\beta K^{cdi}K_{cd}{}^j\phi_i\widetilde{\nabla}^a\phi_j + 2\beta K^{cdi}\widetilde{\nabla}^a K_{cd}{}^j\phi_i\phi_j, \end{aligned} \tag{44}$$

where we can notice that this corresponds exactly to the two-form obtained taking the variation of the symplectic potential, Eq. (43).

Thus, we can notice that in string theory, the [DNG] action with the [GB] topological term has indeed a physically relevant contribution on the symplectic current found in Eq. (44), due to the proportional terms to the parameter β. If the parameter β vanishes, we obtain the symplectic current found for [DNG] action (Cartas-Fuentevilla, 2002; Escalante, 2004).

On the other hand, if we take $\sigma = 0$ and considering the case of string theory in the Eq. (14), the dynamics vanishes, and we would not find apparently any physical motivation for including the [GB] topological term in any action describing strings, however, in this paper the proportional terms to the parameter β in the Eqs. (19), (20), and (44) do not vanish. In this manner, though there is no dynamics of the system but we have a nontrivial deformation dynamics (see Eq. (20) with $\sigma = 0$), and we can construct a nontrivial symplectic structure that lives in the transition of the regimes quantum and classical, that we will utilize in futures works.

It is important to notice that the treatment presented in Cartas-Fuentevilla (2004) is incomplete, because the exterior derivative employed is not correct. However, this work is consistent with the results obtained by the adjoint operators method, Eq. (37), and by the variation of the symplectic potential, Eq. (43).

With the previous results, we can define a two-form on Z in terms of Eq. (44), that will be our symplectic structure

$$\omega \equiv \int_\Sigma \sqrt{-\gamma}\, j^a d\Sigma_a = \int_\Sigma \delta\Psi^a d\Sigma_a, \tag{45}$$

were Σ is a Cauchy surface for the configuration of the string.

We can see that ω is an exact two-form and in particular is closed as δ is nilpotent, and this is

$$\delta\omega = \int_\Sigma \delta(\delta\Psi^a)\, d\Sigma_a = \int_\Sigma \mathbf{D}(\mathbf{D}\Psi^a)\, d\Sigma_a = 0. \tag{46}$$

Now, we will prove that the symplectic structure, found in Eq. (44), is gauge invariant. For this purpose, we observe the degenerate directions on the phase space associated with the gauge transformations of the theory, such degenerate directions will be associated with spacetime infinitesimal diffeomorphisms

$$X^{\mu} \rightarrow X^{\mu} + \delta X^{\mu}. \tag{47}$$

Thus, we shall show that our symplectic structure ω is invariant under the transformation given in Eq. (47). For this, we use the fact that ω is given in terms of the fields $\phi^i = n^i{}_{\mu}\delta X^{\mu}$ and under the transformation Eq. (47), is invariant, that is

$$\phi^i = n^i{}_{\mu}\delta[X^{\mu} + \delta X^{\mu}] = \phi^i, \tag{48}$$

where the second term vanishes because δ is nilpotent; in this manner, we have shown that ω is a gauge invariant.

It is important to notice that δX^{μ}, in the definition of ϕ^i, physically represents the infinitesimal spacetime deformations of the embedding, whereas δX^{μ} in the transformation Eq. (47) is a spacetime diffeomorphism (Cartas-Fuentevilla, 2002).

Therefore, ω is a nondegenerate two-form on Z for [DNG-GB] string theory.

6. CONCLUSIONS AND PROSPECTS

As we have seen, although in string theory the [GB] topological term that we have added to the [DNG] action does not contribute to the dynamics of the system, it has a nontrivial contribution on the Witten covariant phase space, by identifying a symplectic potential as in Escalante (2004), and using a covariant description of the canonical formalism, which is completely unknown in the literature. Taking this into account, we have constructed a covariant and gauge invariant geometrical structure for [DNG-GB] strings, from which we shall study, for example, the relevant symmetries of the system and construct the corresponding Poisson brackets. It is important to mention that the quantization aspects for [DNG] action in string theory is well known, concretely the solutions for the dynamics are known in an explicit way in the literature, which is crucial in the study of such aspects, but other cases are not considered, for example, the system taken in this paper ([DNG-GB] branes), because it is difficult to solve the equations of motion. However, in the case tried here the equations of motion for [DNG] and [DNG-GB] in string theory are the same (Eq. (18)), and specifically their solutions. In this manner, we can take advantage of this fact and use the solutions known to the equations of motion for [DNG] string, and the results presented here to reveal explicitly the contribution of the topological term on the quantization aspects of the theory under study, which has not been considered in the literature. However, this is a future work.

In addition, it is important to mention that the same treatment presented in this paper is applicable for the First Chern number, and that also is a topological invariant for the worldsheet sweep out a string embedded in a 4-dimensional background spacetime, but this calculation we shall develop in future works when we will require it explicitly.

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